Curve Fitting Algorithms for Modal Analysis

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Abstract: This paper discusses deficiencies of techniques widely adopted in modal analysis for curve fitting and introduces a new direct simultaneous modal approximation method, significantly improving the accuracy of modal parameters reconstructed from experimental measurements. The method employs a Newton iteration technique to approximate simultaneously all eigenvalues and eigenvectors of the structure. A major advantage of the method is that it minimises the global square error for all available data, enabling accurate reconstruction of modal parameters from the data containing significant levels of noise.

1 INTRODUCTION

A common method of predicting the behaviour of real systems in many areas of science and engineering is to use relevant mathematical models. One of the fundamental problems is the verification of such models using data from experimental measurements. In particular, specific parameters of the model have to be found that provide the best possible match between the behaviour of the model and the experimental observations.

This paper deals with the identification of parameters of mathematical models of structures used for their vibration analysis.

There are essentially two types of mathematical models used for the linearized vibration analysis of a structure:

(1) **Spatial** models, which incorporate structural mass, stiffness and damping matrices determined along a selected set of coordinates on the structure. To date, there are no effective methods of determining such matrices on the basis of experimental vibration measurement.
(2) **Modal** model, in which the dynamic characteristics of the structure are described using the eigenvalues (natural frequencies) and eigenvectors (mode shapes) of the structure.

Spatial models are usually obtained as a result of discrete modelling of the system, either analytically or using a Finite Element computer algorithm. Modal models can be obtained either from their spatial equivalents by performing a (numerical) eigenvalue analysis, or from an experiment, by approximating the measured Frequency Response Functions (FRF) by an appropriate set of functions.

The two models outlined above are mathematically equivalent, and if the data is accurate and complete, one model can be obtained from the other with a precision limited only by the accuracy of the numerical computations.

The real problem arises when we attempt to use the experimentally identified modal model (eigenvalues and eigenvectors) to estimate the system matrices. Such a procedure is desirable for many reasons, for example to identify and monitor structural changes (cracks etc..). Because of practical limitations associated with experimental procedures, only a limited number of eigenvalues and eigenvectors can be obtained. Furthermore, experimental vibration measurements contain noise and consequently the modal parameters are reconstructed with errors.

The process of extracting modal properties from the experimental vibration measurement has developed over the years into a specialised engineering discipline called "experimental modal analysis".

The main purpose of this paper is to discuss deficiencies of the existing modal analysis algorithms and introduce a new algorithm, significantly increasing the accuracy of the reconstructed eigenvalues and eigenvectors from noisy experimental measurements.

2 PRINCIPLES OF MODAL ANALYSIS

A discrete (finite dimensional) model of a mechanical system reduced to selected set of n coordinates is described by a linear differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F \qquad (1)$$

where: n - is the number of physical coordinates in the system, $x = (x_1 ... x_n)^T$ - is the n-dimensional vector of physical displacement coordinates, $F = (F_1 ... F_n)^T$ - is the n-dimensional vector of excitation forces along components of x ; m, k and c are the $n \times n$ symmetric mass, stiffness and damping matrices of the reduced discrete system. Matrices m, c and k

constitute the so called spatial model of the structure.

In this paper, we shall consider systems with symmetric mass, damping and stiffness matrices m, c and k. Discrete models with symmetric matrices are very important for two reasons:

(1) they can represent a great majority of mechanical, civil and other linearised systems, when their small vibrations around the equilibrium position are considered
(2) they allow us to reconstruct eigenvalues and eigenvectors of the system from the incomplete measurements of the Frequency Response Functions (FRF) (for example from a single column of the FRF matrix in the best case)

Using a $2n$-dimensional variable

$$\hat{x} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \tag{2}$$

representing a set of coordinates, extended by system velocities, equation (1) is transformed to the following form

$$A\dot{\hat{x}} + B\hat{x} = Q \tag{3}$$

where $A = \begin{pmatrix} c & m \\ m & 0 \end{pmatrix}$; $B = \begin{pmatrix} k & 0 \\ 0 & -m \end{pmatrix}$; $Q = \begin{pmatrix} F \\ 0 \end{pmatrix}$

Note, that since m, c and k are symmetric, A and B are also symmetric. It can be shown that the mass matrix m is positive definite, so all its eigenvalues are positive and its inverse m^{-1} exists. Note, that if m^{-1} exists, then A^{-1} also exists.

2.1 Eigenvalues, eigenvectors and their properties

To study free vibration we need to consider equation (3) without external excitation i.e. $Q = 0$.

$$A\dot{\hat{x}} + B\hat{x} = 0 \tag{4}$$

Multiplying both sides of (4) by A^{-1} we have:

$$\dot{\hat{x}} = \Lambda\hat{x} \tag{5}$$

where we denoted $\Lambda = -A^{-1}B = \begin{pmatrix} 0 & I \\ -m^{-1}k & -m^{-1}c \end{pmatrix}$

Suppose Λ has distinct eigenvalues $\lambda_1 ... \lambda_{2n}$ and corresponding eigenvectors $U_1 ... U_{2n}$. Then we have $U^{-1}\Lambda U = \left[\widehat{\Lambda}\right]$,viz: $\left[\widehat{\Lambda}\right]$ is a diagonal matrix containing all system eigenvalues and $U = (U_1 ... U_{2n})$ is the eigenvector matrix. The upper and lower parts of $\hat{x}$ are related: the upper parts represent displacements and lower parts corresponding velocities. Consequently, the upper and lower parts of each eigenvector U_k are also related: the upper parts of U_k are amplitudes of displacements and the lower parts are amplitudes of the corresponding velocities :

$$U_k = \begin{pmatrix} u_k \\ u_k\lambda_k \end{pmatrix}, \qquad k = 1, .., 2n \tag{6}$$

Choose $\hat{x}_k = U_k \exp \lambda_k t$, $k = 1, ..., 2n$ as a basis of a linear space S. All non-trivial solutions of (3) belong to this linear space. It means, that any free motion of the system, described by equation (3) can be presented as a linear combination of $\hat{x}_k$.

Since matrix Λ is real, its eigenvalues must be real or occur in complex conjugate pairs. In this paper we shall consider only the latter case. Consequently, the corresponding eigenvectors must also occur in complex conjugate pairs. Assume, that all eigenvectors in U are normalized, so that they have a unit norm. Since a real matrix Λ is not symmetric, its eigenvector matrix U is NOT orthogonal. However, when matrices A and B are symmetric, we have the following relationships :

$$U^T A U = \left[\hat{A}\right] \quad \text{and} \quad U^T B U = \left[\hat{B}\right] \tag{7}$$

where $\left[\hat{A}\right]$ and $\left[\hat{B}\right]$ are diagonal matrices and $\frac{\hat{B}_k}{\hat{A}_k} = \lambda_k$, $k = 1...2n$. Define $M_k = \sqrt{\hat{A}_k}$ as the modal masses and $K_k = \sqrt{\hat{B}_k}, k = 1...2n$ as the modal stiffnesses. We have $\lambda_k = \sqrt{\frac{K_k}{M_k}}$. Now define mass normalized eigenvectors Φ_k as follows :

$$\Phi_k = \frac{U_k}{M_k} = \frac{1}{M_k}\begin{pmatrix} u_k \\ u_k\lambda_k \end{pmatrix} = \begin{pmatrix} \phi_k \\ \phi_k\lambda_k \end{pmatrix}, \quad k = 1, .., 2n \tag{8}$$

Considering the above definition in (7) we have the following relationships for the mass normalized eigenvectors Φ:

$$\Phi^T A \Phi = I \quad \text{and} \quad \Phi^T B \Phi = \left[\widehat{\Lambda}\right] \tag{9}$$

We can arrange all eigenvalues in $\left[\widehat{\Lambda}\right]$ so that $\lambda_k = \bar{\lambda}_{k+n}$. Introducing $\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$ we have $\left[\widehat{\Lambda}\right] = \begin{pmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{pmatrix}$

The corresponding eigenvectors will be $U_k = \bar{U}_{k+n}$. Consequently, $\phi_k = \bar{\phi}_{k+n}$, and $n \times 2n$ matrix ϕ can be written as follows:

$$\phi = (\varphi, \bar{\varphi}) \tag{10}$$

where φ is the $n \times n$ matrix of eigenvectors. As we shall see later on, we can identify elements of matrices Λ and φ directly from experimental measurements.

2.2 Forced vibration

Consider equation (3) for a non-zero vector of forces F. Define a new set of modal coordinates z such that

$$\hat{x} = \Phi z. \tag{11}$$

Introducing modal coordinates z into (3) and pre-multiply by Φ^T we have $\Phi^T A \Phi \dot{z} + \Phi^T B \Phi z = \Phi^T Q$. Introducing the relationships in (9) we obtain

$$\dot{z} + \bar{\Lambda} z = \Phi^T Q = R, \tag{12}$$

where $R = \Phi^T Q$ is the modal excitation vector. Equation (12) represents a set of uncoupled ordinary differential equations $\dot{z} + \lambda_s z = R_s$, $s = 1...2n$. Considering harmonic excitation of the form $F = F_0 \exp i\omega t$ we have $Q = Q_0 \exp i\omega t$ and $R = R_0 \exp i\omega t$. A particular solution of the

r-th equation in (12) is therefore $z_r = \frac{P_{r0}}{(i\omega - \lambda_r)} \exp i\omega t$. Introducing a diagonal matrix $\lceil C \rfloor = \begin{bmatrix} \lceil i\omega - \lambda_r \rfloor & 0 \\ 0 & \lceil i\omega - \bar{\lambda}_r \rfloor \end{bmatrix}^{-1}$ we can combine all equations $r = 1...2n$ to obtain $z_0 = \lceil C \rfloor \Phi^T Q_0$. Returning to coordinates $\hat{x}$ according to (11) we have $\hat{x}_0 = \Phi \lceil C \rfloor \Phi^T Q_0$. Solution of the initial set of equations along coordinates x, according to (2) is the upper part of $\hat{x}$. Therefore

$$x_0 = \phi \lceil C \rfloor \phi^T F_0 = \left(\varphi \lceil i\omega - \lambda_s \rfloor^{-1} \varphi^T + \bar{\varphi} \lceil i\omega - \bar{\lambda}_s \rfloor^{-1} \bar{\varphi}^T\right) F_0. \quad (13)$$

Equation (13) represents a **modal model** of the structure, when matrices k, m, and c are symmetric and there are no multiple eigenvalues.

2.3 Frequency Response Function (FRF) matrix H

The frequency response function matrix (FRF) is the matrix containing the responses of the system at coordinate k to a unit force harmonic excitation at coordinate j, in some frequency range $(\omega_{\min}, \omega_{\max})$ where $k, j = 1, ..n$. To calculate elements of H, $h_{kj}(\omega)$ we consider $F_0 = (0...F_j...0)^T$, with $F_j = 1$. We then have:

$$h_{kj}(\omega) = x_{k0}(\omega) = \sum_{s=1}^{n} \left(\frac{\varphi_{ks}\varphi_{js}}{i\omega - \lambda_s} + \frac{\bar{\varphi}_{ks}\bar{\varphi}_{js}}{i\omega - \bar{\lambda}_s} \right) \quad (14)$$

where j - is the excitation coordinate number, k - is the response coordinate number, s - is the mode (eigenvector) number and $\bar{\varphi}$ indicates a complex conjugate of φ. Note, that matrix H is symmetrical, because $h_{kj}(\omega) = h_{jk}(\omega)$.

2.4 Experimental measurements

In practice, it is very difficult and costly to measure all elements of matrix H. Therefore, we need to develop a technique of reconstructing all eigenvectors from only the fragment of H which is available. The simplest case is when a single j-th column H^j is available. This corresponds to the practical situation when the response of the structure due to a single point excitation at coordinate j is measured over a grid of points. Note, that it is always desirable to measure more than a single column of H. (For a method to find the best possible excitation points from a given grid see references [1][2]

Here we shall assume, that we measure elements $\tilde{h}_{kj}(\omega)$ for $k = 1, ..n$ and $j = j_1, ..., j_M$, where functions $\tilde{h}_{kj}(i\omega)$ contain measurement noise.(see references [3][4][5])

3 ALGORITHM REQUIREMENTS

We would like to identify the system parameters (eigenvalues and eigenvectors) from the experimentally measured set of functions $\tilde{h}_{kj}(\omega)$. Examining equation (14), we can see that a single column (or row) of H could theoretically contain all the data necessary to reconstruct all eigenvalues and eigenvectors and therefore also other columns (rows) of matrix H. Note, that it is quite possible (and frequently encountered in practice), that not all modes would have sufficiently large coefficients φ_{js} to be able to be identified. (For a more detailed analysis of this topic see reference [1].)

We emphasize, that the measured data contains errors as well as noise. Therefore we should aim to approximate the noisy experimental data by suitable analytical functions, thereby providing a "filter" for the noise. According to results obtained in the previous sections, the best analytical function to approximate a single element $\tilde{h}_{kj}(\omega)$ of the real system should be of the form of equation (14).

In order to obtain a good approximation for the system eigenvalues and eigenvectors, we have to obtain the best **simultaneous** approximation for all measured functions $\tilde{h}_{kj}(\omega)$ - elements of the FRF matrix $\tilde{H}$. Let us consider a situation, where we reconstruct all possible eigenvectors from columns $j_1....j_M$ of matrix $\tilde{H}$. We can define the error of approximation ε_{kj} for a single function $\tilde{h}_{kj}$ as follows:

$$\varepsilon_{kj}(\omega) = \tilde{h}_{kj}(\omega) - \sum_{s=1}^{S} \left(\frac{\varphi_{ks}\varphi_{js}}{i\omega - \lambda_s} + \frac{\bar{\varphi}_{ks}\bar{\varphi}_{js}}{i\omega - \bar{\lambda}_s} \right) \quad (15)$$

Optimal λ_s (eigenvalues) and φ_{ks} (eigenvectors) are obtained when the following error function is minimised :

$$\min_{(\lambda_s, \varphi_{ks})} \sum_{\omega=\omega_{\min}}^{\omega=\omega_{\max}} \left(\sum_{j=j_1}^{j=j_M} \left(\sum_{k=1}^{n} W_{kj}(\omega) \varepsilon_{kj}^2(\omega) \right) \right) \quad (16)$$

where $W_{kj}(\omega)$ is an optional weighting function, introduced to allow user choice of setting priorities in the approximation process. Note, that all eigenvalues λ_s as well as all eigenvectors φ_{ks} in the above formulation are **common** for all measured functions $\tilde{h}_{kj}(\omega)$. When a **global minimum** of the above error function (16) is found, all eigenvalues λ_s as well as all eigenvectors φ_{ks} are optimal in the sense of the L_2-norm. When more than a single column of $\tilde{H}$ is available,($M > 1$) there is data redundancy for eigenvectors φ_{ks} enabling their determination with better accuracy.

Mathematically, the least square problem defined above is not simple, because we seek λ_s and φ_{ks} **simultaneously**, which makes the problem non-linear. Standard methods of linear algebra are not directly applicable. One of the possible solutions is to fix all λ_s, which enables determination of φ_{ks} by a standard linear least square method. Unfortunately, in this approach, further iterations in a multi-dimensional space of λ_s $s = 1, .., S$ are further required, which makes the algorithm numerically inefficient.

4 EXISTING ALGORITHMS

Practical "curve fitting" methods for modal analysis, incorporated in the "state of the art" commercial software use a two step approach to determine modal properties from experimentally obtained FRFs.:

(1) estimation of eigenvalues
(2) estimation of eigenvectors, by approximating each single function $\tilde{h}_{kj}(\omega)$ or a linear combinations of selected functions $\tilde{h}_{kj}(\omega)$ at a time.

Such an approach **does not** minimize the least square error defined in the previous section (equation (16)). As a consequence, eigenvalues λ_s as well as eigenvectors φ_{ks} determined from such a process are frequently far from optimal. It is well known that even small errors in the system eigenvalues lead to significant errors for the corresponding eigenvectors.

In the previous section we noted, that if the **true** system eigenvalues are known, then the corresponding eigenvectors can be found by a linear least square method based on (16). Therefore, any criticism of the currently used methods should contain a detailed analysis of the methods used for performing step 1) above. Most "advanced" methods of searching for "global" eigenvalues used today are based on the fitting of rational polynomials. By performing the addition of all fractions in (14), any element of the FRF matrix $h_{kj}(\omega)$ can be presented in a form of a ratio of two complex polynomials $P(\omega)$ and $Q(\omega)$: $h_{kj}(\omega) = P_{kj}(\omega)/Q(\omega)$. One can formulate the error function $e_{kj}(\omega)$ as follows:

$$e_{kj}(\omega) = \tilde{h}_{kj}(\omega) - P_{kj}(\omega)/Q(\omega) \qquad (17)$$

and search for polynomial coefficients minimizing $\sum_{k,j,\omega} e_{kj}^2$. Since such a least square problem is non-linear and there are no efficient methods to solve it, the following technique is commonly used. Equation (17) is multiplied by $Q(\omega)$ which gives:

$$E_{kj}(\omega) = e_{kj}(\omega)\,Q(\omega) = Q(\omega)\,\tilde{h}_{kj} - P_{kj}(\omega) \qquad (18)$$

It is quite easy to find such coefficients of polynomials $P_{kj}(\omega)$ and $Q(\omega)$ which minimize the error function $\sum_{k,j,\omega} E_{kj}^2(\omega)$, because the problem is now **linear**. Then, roots of the polynomial $Q(\omega)$ can be found, which are the eigenvalues of the system, because $Q(\omega)$ is precisely the common denominator for all fractions. All is fine, except that the **wrong error function** has been minimised. It is possible to improve the quality of approximation somewhat, by introducing the error weighting function $WE(\omega)$. Unfortunately, the best weighting function happens to be the very polynomial $Q(\omega)$ we have to find. For this reason, the above procedure yelds reasonable results, only when the measured function $\tilde{h}_{kj}(\omega)$ contains very little noise.

The major shortcomings of a rational polynomial approach can be summarised as follows:

(1) The rational polynomial procedure loses its reliability as the level of measurement noise increases. Polynomials seem to follow the noise "wrinkles" on the FRF function and do not provide a robust modal filter.

(2) The procedure is formulated for a single function $\tilde{h}_{kj}(\omega)$, which may not give the best approximation for true system eigenvalues which should be common to all FRFs. An improved version of the method approximates a linear combination of all available functions $\sum_{k,j} a_{kj}\tilde{h}_{kj}(\omega)$. This approach is also deficient, despite the fact that in some cases it can produce quite good results. Its deficiency arises from the fact, that by adding all fractions we have no guarantee that some of them would not simply cancel out, leaving little or no information about the corresponding mode. (Consider for example a symmetric mode shape, for which the sum of all fraction numerators is zero.)

(3) Numerical errors increase significantly with the number of modes. For example considering a system with 20 modes requires evaluation of complex polynomials of 40-th order. Numerical generation and evaluation of roots of such polynomials introduces significant numerical errors even when all computations are performed using extended numerical precision.

(4) Rational polynomial approximation has to be applied to subsets of the experimental data containing only a few modes (approximation is performed considering segments of the available frequency range). Such a procedure does not produce the optimal approximation for all available data simultaneously.

(5) The method does not provide satisfactory approximation accuracy in the case of multiple (or very close) eigenvalues

(6) It is very difficult to control the domain for the eigenvalue approximation. For example, polynomial roots (eigenvalues) with positive real parts (corresponding to an unstable mechanical system) frequently appear as a result of measurement noise and limited numerical accuracy. Similarly, roots (eigenvalues) without complex conjugates, which have no physical sense, may appear as a result of computations. In practice, such roots are simply eliminated from consideration without any analysis.

From the above discussion, it is quite clear, that a two step approach described above (currently used by all commercial modal analysis software packages) is not adequate, especially when the measured data contains noise. This represents significant practical limitation, since there are many structures (buildings, bridges for example) for which a "noise free" data is almost impossible to obtain.

5 DIRECT SIMULTANEOUS MODAL APPROXIMATION

In our algorithm we aim for **direct simultaneous approximation** of eigenvalues and eigenvectors from all measured functions $\tilde{h}_{kj}(\omega)$. We use a Newton iteration method [6] to search for the entire set of λ_s $s = 1, ..., S$ and φ_{ks} providing minimisation of the error function (16). Newton iteration method yields very fast convergence, provided we know a reasonable initial approximation for the system eigenvalues λ_s $s = 1, ..., S$ and eigenvectors φ_{ks}.

When the measured data contains no noise (for example $\tilde{h}_{kj}(\omega)$ is computed numerically), the eigenvalues and eigenvectors of the system can be reconstructed to within 7 significant digits. (This statement has been substantiated by reconstructing 9 eigenvalues and eigenvectors from an example numerically generated data, starting from initial approximation accurate to within 5%). As the level of noise increases, the accuracy of the eigenvalue approximation decreases, the relationship being almost linear.

When the measured data $\tilde{h}_{kj}(\omega)$ contains very little or no noise, the rate of convergence increases exponentially as the

distance to a local optimum decreases. As the level of noise in the data increases, the rate of convergence becomes slower. Since the rate of convergence of the algorithm improves with the accuracy of initial estimates, it is important to ensure, that such estimates are as accurate as possible, otherwise, the Newton iteration algorithm for the entire set of data can take a long time to converge. Preliminary estimates for λ_s $s = 1, ..., S$ are obtained from a smoothed estimate of the global kinetic energy of the system. This method gives λ_s $s = 1, ..., S$ with accuracy better than 5%. To improve this accuracy, approximation of a single FRF function (usually a driving point(s) FRF $\tilde{h}_{jj}(i\omega)$) is performed via the Newton iteration technique using such preliminary estimates. This is quite an efficient way for improving the accuracy of the initial estimates for λ_s, since the dimension of the problem is much smaller than for all the available data, and only a few Newton iterations are normally required. In the next step, the initial values for λ_s , $s = 1, ..., S$ are entered to function (16) and the corresponding estimates are computed for the system eigenvalues using a linear least square approximation procedure. Finally, initial estimates for eigenvalues and eigenvectors are used to initiate the Newton iteration process, minimizing the L_2 -norm (16) for the entire set of data.

It should be pointed out, that λ_s and φ_{ks} obtained as a result of such calculations are the best approximations of the true eigenvalues and eigenvectors of the structure, only if the computed local minimum of the function (16) is the global minimum. Newton iteration process always finds a **local** minimum, such that no improvement can be made by any small perturbation of eigenvalues and eigenvectors. In many cases we can state (from physical considerations) that we indeed have obtained the global optimum. However, we do not have the proof that we in fact found the global minimum. The problem of proving that the global minimum has been achieved is very important, but it leads to rather complicated mathematical analysis. Difficulties arise because we have to investigate the global geometry of an infinite dimensional manifold which is in a sense an "infinite dimensional Grassmanian". Points of this manifold are $2l$-dimensional submanifolds L_Λ in $L^2_{[\omega_{min},\omega_{max}]}$. The submanifold L_Λ is the linear envelope of the following fractions :

$$\left\{ \frac{1}{i\omega - \lambda_s} \, , \, \frac{1}{(i\omega - \lambda_s)(i\omega - \bar{\lambda}_s)} \right\} \qquad (19)$$

,where $\Lambda = \{\lambda\}_{l=1}^{n}$ is the set of eigenvalues of the transfer matrix; the reader can find further details in [1].

5.1 Residual modes

In practice, the frequency range for measurement of $\tilde{h}_{kj}(\omega)$ is limited. This means, that modes can exist outside the considered frequency range. In our algorithm we add a number of residual modes outside the frequency range of the data if required, in the stage of estimating the initial values for λ_s , $s = 1, ..., S$. In most cases adding just one or two residual modes is sufficient to take into account the effects of all neglected modes outside the frequency range of the measurement. (It should be noted, that eigenvalues and eigenvectors obtained for such modes could contain significant errors).

6 APPLICATION

The newly developed Direct Simultaneous Modal Approximation Method has been applied to extract eigenvalues and eigenvectors of a reinforced concrete bridge near Rutherglen in northern Victoria, Australia, from a single column of FRF's obtained as a result of modal testing experiment. The bridge was excited by a hydraulic shaker and its response was measured along a grid of 80 coordinates. Only 10 accelerometers were available, so this set was relocated several times to measure the bridge response along all coordinates. 15 averages have been performed for each measurement of the FRFs. It is impossible to depict all 80 FRFs , so we have selected just two, representative variations from this set. Fig.1 demonstrates the approximation of the measured Frequency Response Function (FRF) of the driving point. To illustrate the quality of the Direct Approximation Method we have shown in Fig.2 the FRF with the **worst** approximation error from the total set. Figures 1 and 2 contain two plots each. The upper plot is the amplitude of the corresponding FRF($|h_{kj}(i\omega)|$) and the bottom plot is the phase $(\tan^{-1}[\mathrm{Im}(h_{kj}(i\omega)) / \mathrm{Re}(h_{kj}(i\omega))])$

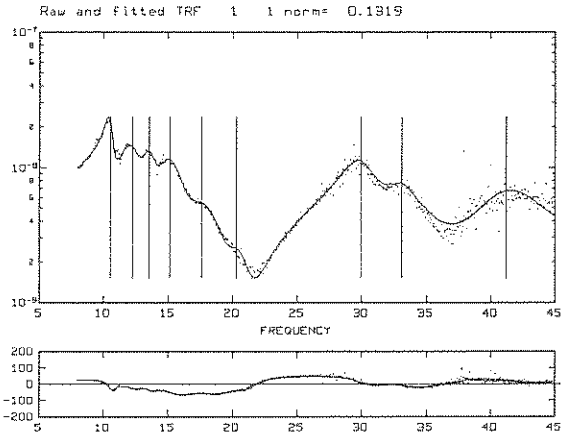

Fig.1. Frequency Response Function of the driving point

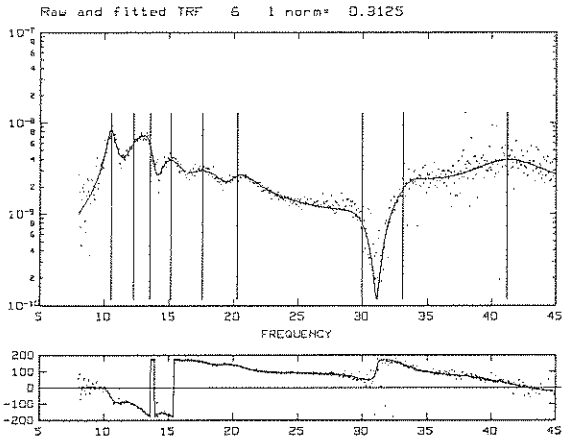

Fig.2. Most "noisy" FRF from the set of 80 measurements

As can be seen from Fig.1 and Fig.2 all FRFs in the measured set contain a significant level of noise and Direct Simultaneous Modal Approximation Method provides a very effective way of filtering it out. Note the exceptional accuracy of the approximation the FRF phase (bottom plots) in both Figures 1 and 2.

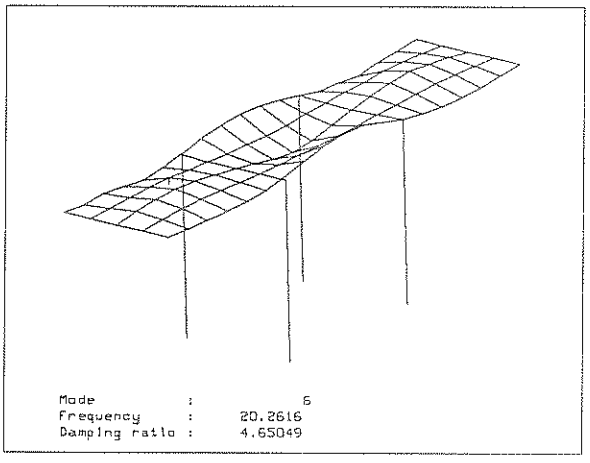

Fig.3. Sixth mode of the concrete bridge at 20.2616 Hz reconstructed from experimental measurements.

The first 9 modes of the bridge have been identified in the frequency range from 5 to 45 Hz. Fig.3 demonstrates mode 6 at 20.26 Hz of the bridge, and also illustrates the measurement grid. A short vertical line at one of the grid points indicates the position of the shaker. We have chosen this mode to demonstrate some of the advantages of the high quality Direct Simultaneous Modal Approximation in modal analysis. It can clearly be seen from the shape of the mode and also from the FRF in Fig.1, that the shaker is located very close to the **nodal line** for mode 6. Because of this unfortunate position of the shaker, contribution of mode 6 to the system response is minimal. To make things worse such a small contribution is embedded in the significant measurement noise. Despite this, using the Direct Simultaneous Modal Approximation algorithm described in this paper, the mode and its shape was successfully reconstructed to high precision. We would like to point out, that we have also tried to use a conventional (2-step) method based on rational polynomials, (before the DSMA algorithm was available) and we were not able to succeed in obtaining this particular mode.

7 CONCLUSIONS

(1) It was demonstrated, that existing "curve fitting" algorithms used for modal analysis, **do not** estimate eigenvalues and eigenvectors simultaneously and therefore cannot in general provide an optimal approximation of the Frequency Response Functions as well as the modal parameters of a structure. Existing algorithms are very difficult to use when the measurement data contains significant levels of noise.

(2) The Direct Simultaneous Modal Approximation Method (DSMA) described in this paper uses a Newton iteration technique to provide simultaneous approximation of the eigenvalues **and** eigenvectors from all available data, that is **optimal** in the sense of the least squares error.

(3) It was shown using the example modal analysis performed for a concrete bridge, that the new DSMA method is superior to the conventional method of experimental modal analysis when the measured data contains noise.

(4) The method finds a **local** minimum for the least squares error function, i.e finds a set of eigenvalues and eigenvectors which cannot be improved by small perturbations, and for this reason the estimation of the initial approximation is important. Further research is required to design algorithms that guarantee the global minimum of the error function.

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